

A statistical mechanics of an oscillator associative memory with scattered natural frequencies

Toru Aonishi, Koji Kurata

Department of Systems and Human Science, Osaka University, 1-3, Toyonaka, Osaka 560, Japan

Masato Okada

ERATO Kawato Dynamic Brain Project, 2-2 Hikaridai, Seika-cho, Soraku-gun, Kyoto 619-02, Japan

(February 1, 2008)

Abstract

Analytic treatment of a non-equilibrium random system with large degrees of freedoms is one of most important problems of physics. However, little research has been done on this problem as far as we know. In this paper, we propose a new mean field theory that can treat a general class of a non-equilibrium random system. We apply the present theory to an analysis for an associative memory with oscillatory elements, which is a well-known typical random system with large degrees of freedoms.

PACS numbers: 87.10.+e, 05.90.+m, 89.70.+c

The analytic treatment of a non-equilibrium random system with large degrees of freedoms is one of most important problems of physics. However, little research has been done on this problem as far as we know. In this paper, we propose a new mean field theory that can treat a general class of a non-equilibrium random system. We apply the present theory to an analysis for an associative memory with oscillatory elements, which is a well-known typical random system with large degrees of freedoms.

Historically, there are two important studies regarding the non-equilibrium systems with large degrees of freedoms that cannot be treated by conventional equilibrium statistical mechanics, since we can not define the Lyapunov function with “bottoms”. Kuramoto [4] theoretically analyzed the mutual entrainment of uniformly coupled oscillators with scattered natural frequencies (Kuramoto theory). His model corresponds to a mean field model of a ferromagnet in equilibrium statistical mechanics. Kuramoto utilized the spilit of statistical mechanics, namely, a notion of macroscopic order-parameters, to investigate his non-equilibrium system with large degrees of freedoms. Daido [5] numerically analyzed the quasi-entrainment of randomly coupled oscillators with scattered natural frequencies. His model corresponds to the Sherrington-Kirkpatrick model of a spin glass [2] in equilibrium statistical mechanics. A mean field theory should be developed for non-equilibrium random systems *with* frustration that follows the history of mean field theories in the equilibrium statistical mechanics. However, this kind of non-equilibrium random system has not yet been theoretically analyzed (see [6] for the random system *without* frustration). This is a famous “open problem” in physics [7]. That is reason why we have proposed the present theory.

On the other hand, the Lyapunov function can not be defined in spin-like systems with the non-monotonic reaction function either. In the self-consistent signal-to-noise analysis (SCSNA) [10], the notion of a macroscopic order parameteris also introduced to analyze these frustrated spin-like systems, which cannot be treated by conventional equilibrium statistical mechanics. Note that the results of applying the SCSNA to simple random spin systems [10,11], i.e., the Sherrington-Kirkpatrick model and the Hopfield model [1], coincide with those of the replica theories [12,13].

In this paper, we propose a new theoretical framework for an oscillator associative memory model with scattered natural frequencies in memory retrieval states. This system can be considered as a typical example of non-equilibrium random systems with large degrees of freedoms. The present theory makes a bridge between the SCSNA and the Kuramoto theory. Using the same procedure, we can easily treat a glass oscillator system [5]. Our theory is reduced to the Kuramoto theory in the finite loading case. When all oscillators have uniform natural frequencies, our theory coincides with the previously proposed theories [9,3] in the equilibrium statistical mechanics for an XY spin system.

The mutual entrainment is an important notion of a non-equilibrium system with large degrees of freedoms. In uniformly coupled oscillators, there is a unique stable state, the ferromagnetic phase in the phase space. On the other hand, in frustrated systems, there are many stable states in the phase space. We need to elucidate the properties of the mutual entrainment in each stable phase (ferromagnetic phase and glass phase). Our new theory describes a phenomenon of the mutual entrainment in the ferromagnetic phase (memory retrieval). Thus, we numerically study a degree of the mutual entrainment in the glass phase (spurious memory retrieval). It is numerically shown in this paper that almost all

oscillators synchronize under memory retrieval, but desynchronize under spurious memory retrieval when setting optimal parameters. Thus, it is possible to determine whether the recalling process is successful or not using information about the synchrony/asynchrony.

In general, when the coupling is sufficiently weak, the high-dimensional dynamics of a coupled oscillator system can be reduced to the phase equation [4,8]. Let us consider the following simplified model,

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^N J_{ij} \sin(\phi_j - \phi_i + \beta_{ij}), \quad (1)$$

where N is the total number of oscillators, ϕ_i is the phase of the i -th oscillator, and ω_i is the natural frequency assumed to be randomly distributed over the whole population with a density denoted by the symmetric distribution $g(\omega)$, i.e., $g(\omega) = g(-\omega)$. Note that the average of ω_i may be set to zero without loss of generality. The theory presented below can be easily extended to treat the system with other any distribution $g(\omega)$. J_{ij} and β_{ij} denote an amplitude of a synaptic weight and a synaptic delay, respectively. In order to investigate the nature of frustrated non-equilibrium systems, we have selected the following generalized Hebb learning rule [9] to determine J_{ij} and β_{ij} ,

$$C_{ij} = J_{ij} \exp(i\beta_{ij}) = \frac{1}{N} \sum_{\mu=1}^p \xi_i^\mu \bar{\xi}_j^\mu, \quad \xi_i^\mu = \exp(i\theta_i^\mu), \quad (2)$$

as their typical example. $\{\theta_i^\mu\}_{i=1,\dots,N,\mu=1,\dots,p}$ are the phase patterns to be stored in the network and are assigned to random numbers with a uniform probability in $[0, 2\pi]$. Here, we define a parameter α (loading rate) such that $\alpha = p/N$. In the equilibrium limit of this model, that is, $g(\omega) = \delta(\omega)$, the storage capacity given by $\alpha_c = 0.038$ [3].

We put $s_i = \exp(i\phi_i)$ for the sake of simplicity. The order parameter m^μ , which measures the overlap between system state s_i and embedded pattern ξ^μ , is defined as

$$m^\mu = \frac{1}{N} \sum_{j=1}^N \bar{\xi}_j^\mu s_j. \quad (3)$$

We obtain order parameter equations of the present system by applying the following manipulations.

- First, assuming a self-consistent local field for each of oscillators, a distribution of s_i under $g(\omega_i)$ is formally derived by the Kuramoto theory.
- Second, we estimate the contribution of randomness, that is, the uncondensed patterns in the present case by the SCSNA, and determine the local field in a self-consistent manner.
- Finally, the order parameter equations are obtained using the self-consistent local field.

A detailed derivation of the present theory will be discussed elsewhere. Here, we assume $m^1 = O(1)$ and $m^\mu = O(1/\sqrt{N})$ for $\mu > 1$ (uncondensed patters). Then, the following two dimensional equations for the order parameters are obtained,

$$m = \langle \langle X(x_1, x_2, \xi) \rangle \rangle_{x_1, x_2, \xi}, \quad (4)$$

$$U = \langle \langle F_1(x_1, x_2, \xi) \rangle \rangle_{x_1, x_2, \xi}, \quad (5)$$

where $\langle \langle \cdots \rangle \rangle_{x_1, x_2, \xi}$ is taken to mean the Gaussian average over x_1, x_2 and condensed pattern ξ^1 , $\langle \langle \cdots \rangle \rangle_{x_1, x_2, \xi} = \langle \int \int D x_1 D x_2 \cdots \rangle_\xi$. The pattern superscripts 1 of m are omitted for brevity. The self-consistent mean field \tilde{h} , X , F_1 and the Gaussian measure $D x_1 D x_2$ are expressed as follows,

$$D x_1 D x_2 = \frac{d x_1 d x_2}{2 \pi \rho^2} \exp \left(-\frac{x_1^2 + x_2^2}{2 \rho^2} \right), \quad (6)$$

$$\rho^2 = \frac{\alpha}{2(1-U)^2}, \quad \tilde{h} = \xi m + x_1 + i x_2, \quad (7)$$

$$X(x_1, x_2, \xi) = \tilde{h} \int_{-1}^1 d x g(|\tilde{h}|x) \sqrt{1-x^2}, \quad (8)$$

$$F_1(x_1, x_2, \xi) = \int_{-1}^1 d x \left(g(|\tilde{h}|x) + \frac{|\tilde{h}|}{2} x g'(|\tilde{h}|x) \right) \sqrt{1-x^2}. \quad (9)$$

Here, U corresponds to the susceptibility, which measures the sensitivity to external fields. A distribution of resultant frequencies $\bar{\omega}$ in the memory retrieval state, which is denoted as $p(\bar{\omega})$, becomes

$$p(\bar{\omega}) = r \delta(\bar{\omega}) + \int D x_1 D x_2 \frac{g \left(\bar{\omega} \sqrt{1 + \frac{|h|^2}{\bar{\omega}^2}} \right)}{\sqrt{1 + \frac{|h|^2}{\bar{\omega}^2}}}, \quad (10)$$

$$r = \int D x_1 D x_2 |\tilde{h}| \int_{-1}^1 d x g(|\tilde{h}|x), \quad (11)$$

where r measures the ratio between the number of synchronous oscillators and the system size N . We now consider the relationships between the present theory and the previously proposed theories. For the equilibrium limit, $g(x) = \delta(x)$, we obtain

$$X = \frac{\tilde{h}}{|\tilde{h}|}, \quad F_1 = \frac{1}{2|\tilde{h}|}, \quad p(\bar{\omega}) = \delta(\bar{\omega}), \quad (12)$$

which coincide with the replica theory [3] and the SCSNA [9]. On the other hand, regarding the uniform-system limit, $\alpha = 0$, our theory reproduces the Kuramoto theory as

$$m = m \int_{-1}^1 d x g(|m|x) \sqrt{1-x^2}, \quad (13)$$

$$p(\bar{\omega}) = r \delta(\bar{\omega}) + \frac{g \left(\bar{\omega} \sqrt{1 + \frac{|m|^2}{\bar{\omega}^2}} \right)}{\sqrt{1 + \frac{|m|^2}{\bar{\omega}^2}}}. \quad (14)$$

$$r = |m| \int_{-1}^1 d x g(|m|x). \quad (15)$$

As mentioned before, our theory has made a bridge between the equilibrium-frustrated system and the non-equilibrium-uniform system. In addition, we have presented a systematic way of analytical treatments for the non-equilibrium random systems. If C_{ij} is assigned to random numbers with Gaussian in complex plane, $\text{Re}[C_{ij}] \sim \mathcal{N}(1/N, \alpha/2N)$, $\text{Im}[C_{ij}] \sim \mathcal{N}(0, \alpha/2N)$, the order parameter equation consists with Eq. 4 under constraint $U = 0$. The detail results will be presented elsewhere. We can also treat Daido's glass oscillator system (with real number interaction) [5] using the same procedure.

In the following analyses, we choose a system with $g(\omega) = (2\pi\sigma^2)^{-1/2} \exp(-\omega^2/2\sigma^2)$. Fig. 1(a) shows a phase diagram in the $(|m|, \sigma, \alpha)$ space, which was obtained by numerically solving these order parameter equations. A cross-section of this curved surface at $\sigma = 0$ coincides with the results of the SCSNA [9] and the Replica theory [3]. Furthermore, a cross-section of this curved surface at $\alpha = 0$ is equal to a result of the Kuramoto theory [4]. Thus, our theory bridges the gap between these theories. Fig. 1(b) shows the values of critical memory capacity α_c for various values of σ . α_c decreases monotonically as σ increases. The critical value of σ at $\alpha_c = 0$ is given by $\sigma_c = 0.62$, which coincides with that of the SK theory. Figures 2(a),(b),(c), and (d) display $|m|$ values vs σ for various values of α , where the solid curves are obtained theoretically, and the plots show results obtained by numerical simulation. According to these figures, the theory is in good agreement with the simulation results.

Next, we examined the distributions of the resultant frequencies $\bar{\omega}_i$ over the whole population of oscillators in the memory retrieval state and the spurious memory state. Here, the resultant frequencies $\bar{\omega}_i$ were calculated by using the long time average of $d\phi_i/dt$. The plots in Figures 3(a) and (b) show the values of resultant frequencies $\{\bar{\omega}_i\}_{i=1,\dots,N}$ versus σ , that is, a bifurcation diagram. Fig. 3(a) denotes $\bar{\omega}_i$ distributions in memory retrieval states, and Fig. 3(b) represents those in spurious memory states. These results show that there exists a region σ that satisfies the two conditions: all oscillators mutually synchronize in memory retrieval, and oscillators desynchronize in spurious memory retrieval. Thus, it is possible to determine whether the recalling process is successful or not only using the information about the synchrony/asynchrony, when proper σ is given. In Figures 3(a) and (b), all phase values ϕ_i continued drifting toward negative directions due to an offset of scattered ω_i . In other samples, all ϕ_i continued drifting toward positive directions.

We investigated the effect of the system size N on the $\bar{\omega}_i$ distribution. The solid curves in Fig. 4 indicate the distributions of the resultant frequencies $p(\bar{\omega})$ in Eq. 10. The histograms in Fig. 4 show the results from the numerical simulation. As shown in Figures 4(a)(a')(a''), the theory agrees well with the simulation results in memory retrieval states. As shown in Figures 4(b)(b')(b''), there exists a delta-peak that indicates the mutual entrainment of a large population of oscillators in memory retrieval states. However, in spurious memory states, the distribution of the average frequency is gentle compared to the distribution in memory retrieval states, which indicates the quasi-entrainment of oscillators [5]. According to these figures, we believe the phenomena mentioned before are invariant to the system size N . The reason why all oscillators mutually synchronize in memory retrieval states, but desynchronize in spurious states, is because effective interactions among oscillators in memory states (as in ferromagnetic states [4]) are different from interactions in spurious memory states (as in spin glass states [5]), in which the system is strongly frustrated.

The phase description proposed here can be considered as a minimum model of neural

networks based on oscillatory activities that is mathematically solvable. Chaos neural networks yield rich phenomena as discussed here, but can not be easily analyzed, except with simulations. Since the present analysis corresponds to the replica symmetric approximation, we have noted that it should be extended to the replica symmetry breaking in order to properly treat the spurious states (spin glass states), and this remains for a future work.

In the field of neuroscience, a growing number of researchers have been interested in the synchrony of oscillatory neural activities because physiological evidence of their existence has been obtained in the visual cortex of a cat [14,15]. Much experimental and theoretical research exists regarding the functional role of synchronization. One of the more interesting hypotheses is called *synchronized population coding*, which was proposed by Phillips and Singer. However, its validity is highly controversial [16]. In this paper, we numerically showed the possibility of determining if the recalling process is successful or not using information about the synchrony/asynchrony. If we consider information processing in brain systems, the solvable *toy* model presented in this paper may be a good candidate for showing the validity of a synchronized population coding in the brain, and we believe the present analysis may strongly influence debate on the functional role of synchrony.

The authors thank Ms. M. Yamana, Mr. M. Yoshioka and Dr. M. Shiino for their valuable discussions. This work was partially supported by Grants-in-Aid for Scientific Research in Priority Area (2) No. 07252219, Grants-in-Aid for the Encouragement of Young Scientists No. 0782 and JSPS Research Fellowships for Young Scientists.

REFERENCES

- [1] J. J. Hopfield, Proc. Natl. Acad. Sci. USA, **79**, 2554 (1982).
- [2] D. Sherrington, and S. Kirkpatrick, Phys. Rev. Let. **35**[26], 1792 (1975).
- [3] J. Cook, J. Phys. A: Math. Gen., **22**, 2057 (1989).
- [4] Y. Kuramoto, *Chemical oscillations, waves and turbulence* (Springer-Verlag, 1984).
- [5] H. Daido, Phys. Rev. Let., **68**, 1073 (1992).
- [6] H. Daido, Prog. Theor. Phys. **77**, 622 (1987).
- [7] J. J. Hopfield and A. V. M. Hertz, Proc. Natl. Acad. Sci. USA, **92**, 6655 (1995).
- [8] G. B. Ermentrout, Journal of Mathematical Biology, **6**, 327 (1981).
- [9] K. Okuda, (unpublished).
- [10] M. Shiino and T. Fukai, J. Phys. A: Math. Gen., **25**, L375 (1992).
- [11] M. Okada, T. Fukai and M. Shiino, Phy. Rev. E, **57**, 2095-2103 (1998).
- [12] A. J., Bray, H. Sompolsky and C. Yu, J. Phys. C: Solid State Physics, **19**, 6389-6406. C. (1986).
- [13] Kühn, S., Bös, S., & van Hemmen, J. L. Phys. Rev. A, **43**, 2084 (1991).
- [14] R. Eckhorn, R. Bauer, W. Jordan, M. Brosch, W. Kruse, M. Munk, and H. J. Reitboeck, Biol. Cybern. **60**, 121 (1988).
- [15] C. M. Gray, P. König, A. K. Engel, and W. Singer, Nature, **338**, 334 (1989).
- [16] W. A. Phillips and W. Singer, Behavioral and Brain Science, **20**, 657 (1997).

FIGURES

FIG. 1. (a) Phase diagram in $(|m|, \sigma, \alpha)$ space. (b) σ vs critical memory capacity α_c .

FIG. 2. Values of $|m|$ vs σ (solid curves theoretically obtained, and plots obtained by numerical simulation). (a) $\alpha = 0.0$ ($p = 1$, $N = 1000$). (b) $\alpha = 0.01$ ($N = 2000$). (c) $\alpha = 0.02$ ($N = 2000$). (d) $\alpha = 0.03$ ($N = 2000$).

FIG. 3. Values of resultant frequencies $\{\bar{\omega}_i\}_{i=1,\dots,N}$ vs σ , $N = 2000$ and $\alpha = 0.0315$. (a) Memory pattern retrieval. (b) Spurious memory pattern retrieval.

FIG. 4. Distribution of resultant frequencies state. $\alpha = 0.01$, $\sigma = 0.32$. Solid curves show theoretical results of $p(\bar{\omega})$ in Eq. 10, but delta-peaks at $\bar{\omega} = 0$ are not indicated. Figs. (b)(b')(b'') display detailed distributions of Figs. (a)(a')(a'') at $\bar{\omega} = 0$, respectively. (a)(b) $N = 2000$. 20 trials. (a')(b') $N = 4000$. 12 trials. (a'')(b'') $N = 8000$. 12 trials.

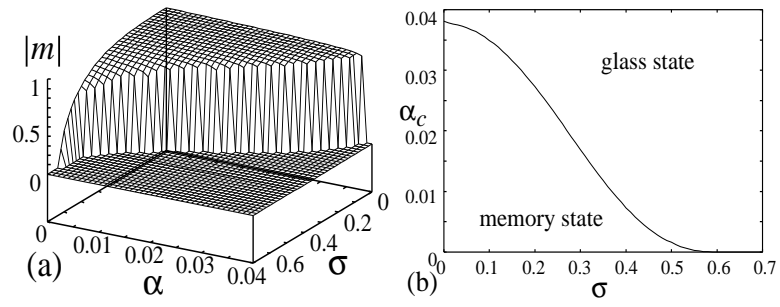


Figure 1
T. Aonishi, K. Kurata and M. Okada

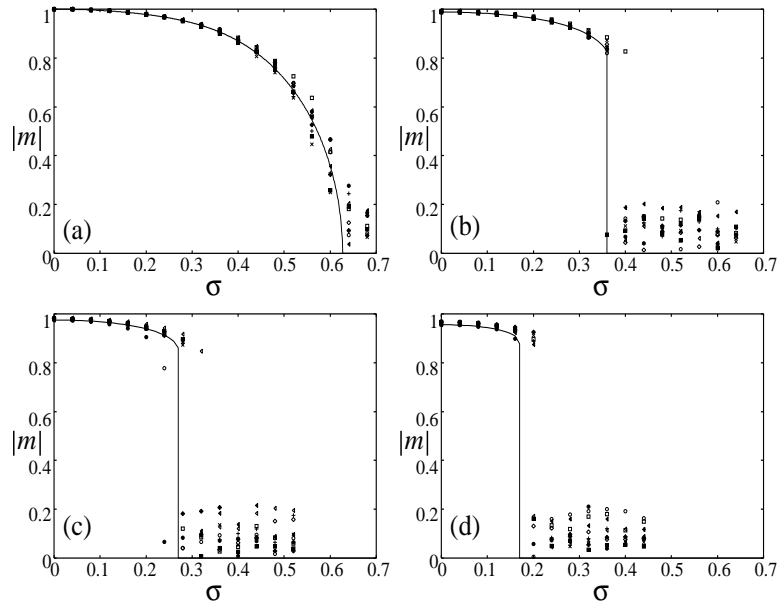


Figure 2

T. Aonishi, K. Kurata and M. Okada

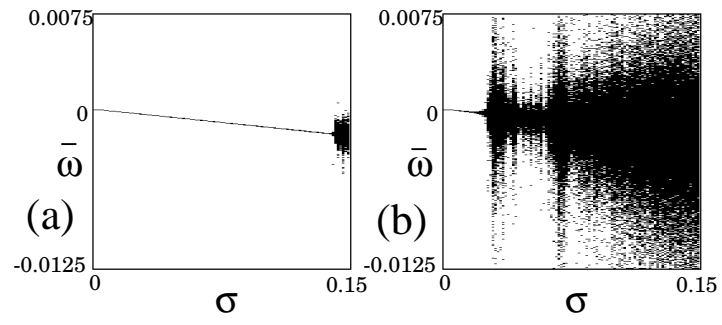


Figure 3
T. Aonishi, K. Kurata and M. Okada

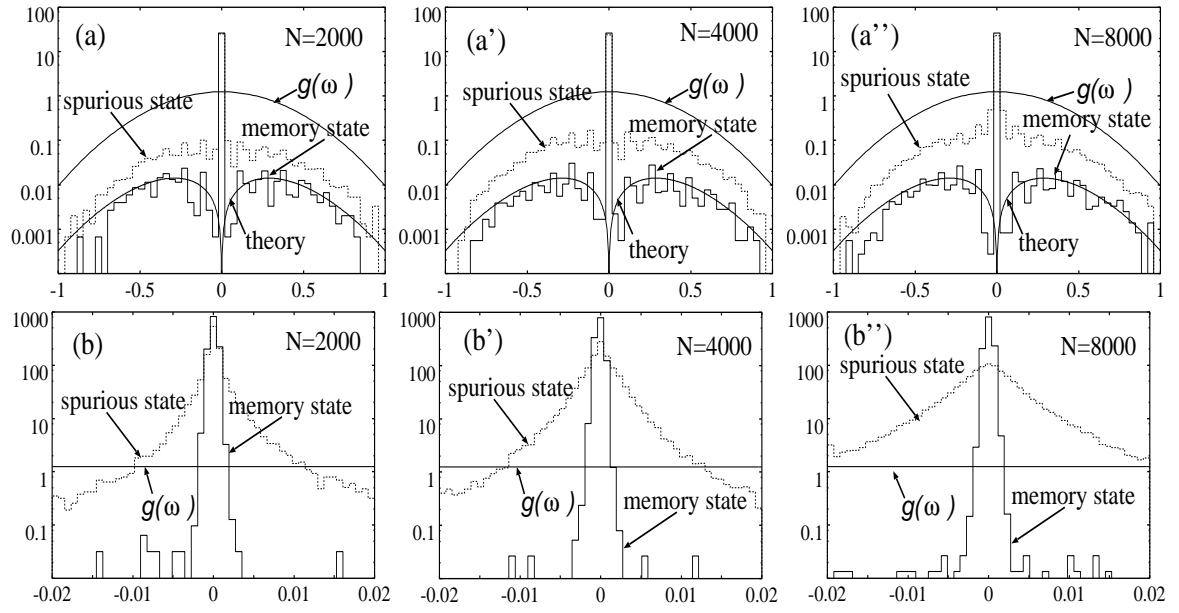


Figure 4

T. Aonishi, K. Kurata and M. Okada